Indian Statistical Institute
B.Math. (Hons.) I Year
Second Semester 2006-07
Mid Semester Examination
Probability Theory II

Time: 3 hrs | Date:09-03-07 | Max. Marks: 50

Instructor: C R E Raja

You may use and quote any result proved in the class room without proof unless you are asked to prove that particular result

Answer any five questions.
Each question has 10 marks.

iid = independent and identically distributed

1. (a) Let X and Y be independent random variables with joint density $f_{X,Y}$ and respective marginal densities f_X and f_Y. Then show that $f(v) = \int_{-\infty}^{\infty} f_X(u)f_Y(v-u)du$ is a density of $X+Y$.

   (b) Find a density of $X+Y$ if X and Y are iid random variables with common density uniformly distributed over $(-1, \ 1)$.

2. (a) If X and Y are independent random variables with normal densities $n(\mu_1, \sigma_1^2)$ and $n(\mu_2, \sigma_2^2)$ respectively, then show that $X+Y$ has normal density $n(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$.

   (b)Let $X_1, X_2, \cdots, X_n$ be independent random variables such that each X_m has normal density $n(\mu_m, \sigma_m^2)$ for $1 \leq m \leq n$. Then show that $X_1 + X_2 + \cdots + X_n$ has normal density $n(\mu, \sigma^2)$ where $\mu = \mu_1 + \mu_2 + \cdots + \mu_n$ and $\sigma^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2$.

   (c) Let $X_1, X_2, \cdots, X_n$ be iid with common normal density. Then show that there are constants A_n and B_n such that $\frac{X_1+X_2+\cdots+X_n-A_n}{B_n}$ has the same density as X_1.

3. Let $U_1, U_2, \cdots, U_n$ be iid random variables with common density uniformly distributed over $(0, \ 1)$. For $1 \leq k \leq n$, let X_k be the kth order statistic and R be the range.

   (a) Find a density of X_k.

(b) Find a joint density of X_1 and X_n.

(b) Find a density of R.

4. (a) Let X be a continuous random variable with density f. Find a density of the random variable $Y = X^2$.

   (b) Let X be a random variable with normal density $n(0, \sigma^2)$. Find a density of $Y = X^2$.

   (c) Let X_1 and X_2 be iid random variables with common normal density $n(0, \sigma^2)$. Find a density of $X^2 + Y^2$.

5. (a) Find density of the F distribution with k_1 and k_2 degrees of freedom.

   (b) Let X and Y be independent random variables with respective densities $n(0, 1)$ and $\chi^2(k)$. Then show that $X/\sqrt{\frac{Y}{k}}$ has t distribution with k degrees of freedom.

6. (a) Let X be uniformly distributed random variable over $(0, \ \ 1)$ and let Y be uniformly distributed over $(0, \ \ X)$. Find the joint density of X and Y.

   (b) Let X and Y be independent random variables with respective gamma densities $\Gamma(\alpha_1, \lambda)$ and $\Gamma(\alpha_2, \lambda)$. Let $Z = X + Y$. Find the conditional expectation of Y given Z.

7. (a) Let $X_1, X_2, \cdots, X_n$ be iid random variables having common exponential density with parameter λ. For $1 \le i \le n$, let $Y_i = X_1 + X_2 + \cdots + X_i$. Find the joint density of $Y_1, Y_2, \cdots, Y_n$.

   (b) Assume $n = 2$ in (a). Find the marginal densities of Y_1 and Y_2 and determine whether Y_1 and Y_2 are independent.